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Periodicity and arithmetic-periodicity in hexadecimal games

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Abstract

We investigate 1-, 2- and some k-digit ($k \geqslant 3$) hexadecimal games with the help of a new arithmetic-periodicity theorem. We also note that not all hexadecimal games are periodic or arithmetic-periodic.

Keywords: Combinatorial game; Hexadecimal; Impartial game; Periodicity; Regularities; Taking-and-breaking

1. Introduction

A *Taking-and-Breaking* game [3] is an impartial, combinatorial game, played with heaps of beans on a table. A move for either player consists of choosing a heap, removing a certain number of beans from the heap and then possibly splitting the remainder into several heaps, the winner is the player making the last move. The number to be removed and the number of heaps that one heap can be split into, is given by the rules of the game.

The rules for an *hexadecimal* game are found in the hexadecimal code $\mathbf{0.d}_1\mathbf{d}_2\ldots\mathbf{d}_u$, where $0 \leqslant \mathbf{d}_i \leqslant 15$. We use the usual letters A, B, C, D, E, F for the numbers 10 through 15, respectively. If $\mathbf{d}_i = 0$ then a player cannot take i beans away from a heap. If $\mathbf{d}_i = \delta_3 2^3 + \delta_2 2^2 + \delta_1 2^1 + \delta_0 2^0$ where δ_j is 0 or 1, a player can remove i beans from

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the heap provided he leaves the remainder in exactly j heaps for some j with $\delta_j = 1$. If, for all i, $0 \leqslant \mathbf{d}_i \leqslant 7$ then this is called an *octal* game. This restriction allows for a heap to be split into no more than 2 heaps. A *subtraction* game has $d_i \in \{0, 3\}$, i.e. a player can remove beans but cannot split the heap.

The followers of a game are all those positions which can be reached in one move. The minimum excluded value of a set S is the least non-negative integer which is not included in S and is denoted by $\text{mex}(S)$. The *nim*-value of an impartial game G, denoted by $\mathscr{G}(n)$, is given by

$$\mathscr{G}(G) = \text{mex}\{\mathscr{G}(H) \mid H \text{ is a follower of } G\}.$$

The values in the set $\{\mathscr{G}(H) \mid H \text{ is a follower of } G\}$ are called *excluded values* for $\mathscr{G}(G)$. An impartial game G is a previous player win, i.e. the next player has no good move, if and only if $\mathscr{G}(G) = 0$.

The *nim-sum* of two non-negative integers is the Exclusive or (XOR), written $\oplus$, of their binary representations. It can also be described as adding the numbers in binary without carrying. A game G is the *disjunctive* sum of games H and K, written $G = H + K$, if, on each turn, the players must choose one of H and K and make a legal move in that game. From the theory of impartial games (see [3] or [5]) *if* $G = H + K$, *then* $\mathscr{G}(G) = \mathscr{G}(H) \oplus \mathscr{G}(K)$.

Taking-and-Breaking games are examples of disjunctive games—choose one heap and play in it. To know how to play these game well, it suffices to know what the nim-values are for individual heaps. For a given game G, let $\mathscr{G}(i)$ be the game played with a heap of size i. We define the $\mathscr{G}$-sequence for a Taking-and-Breaking game to be the sequence $\mathscr{G}(0), \mathscr{G}(1), \mathscr{G}(2), \dots$.

For subtraction games, in [1] it is shown that the sequence of nim-values of games with small subtraction sets can have long periods. Specifically, that game with subtraction set $\{s, 4s, 12s + 1, 16s + 1\}$ has a period length of $56s^3 + 52s^2 + 9s + 1$ for $s = 1, 2, \dots, 26$.

A compendium of periods and results for octal games can be found in [3]. Gangolli and Plambeck [7] found the periods of **0.127**, **0.16**, **0.376** and **0.56**. Flammenkamp has a web site [6] that lists the latest computations and results concerning octal games with at most three digits. At the time this paper was written, some of these calculations have been taken up as far as heap size 2^{35}. In addition, Grundy's game (choose a heap and split it into two unequal heaps) has been analysed [6] to heap size 5×2^{32} and Couples-are-Forever [4] (choose a heap and split it into two but a heap of 2 cannot be split) has been analyzed to heap size 5×10^7 and neither show any sign of becoming periodic.

Examples of hexadecimal games were known that exhibit one of two types of 'periodic' behavior. The first is normal *periodicity*, i.e. there exists N and p such that $\mathscr{G}(n + p) = \mathscr{G}(n)$ for all $n \geqslant N$. Such periodicity occurs also in subtraction and octal games. The second is *arithmetic-periodicity:* there exists N, p and s such that $\mathscr{G}(n + p) = \mathscr{G}(n) + s$ for all $n \geqslant N$, where s is called the *saltus*. For example, the $\mathscr{G}$-sequence for **0.137F** is $0, 1, 1, 2, 2, 3, 3, \dots$, where $\mathscr{G}(2m - 1) = \mathscr{G}(2m) = m$ for $m \geqslant 1$. In this case, the saltus is 1 and the period length is 2 or, $\mathscr{G}(n+2) = \mathscr{G}(n) + 1$ for $n \geqslant 1$. A Taking-and-Breaking game is *split arithmetic periodic*, *periodic regular* or *sapp* *regular* for short, if there exist integers e, s, and p, and a set $S \subseteq \{0, 1, 2, \ldots, p-1\}$ such that

- $\mathcal{G}(\overline{i+p}) = \mathcal{G}(\bar{i})$ for $i > e$ and $(i \bmod p) \in S$,
- $\mathcal{G}(\overline{i+p}) = \mathcal{G}(\bar{i}) + s$ for $i > e$ and $(i \bmod p) \notin S$.

In [9], some octal games, where one heap has a pass move associated with it, are shown to be sapp regular.

Arithmetic-periodicity does not occur in octal or subtraction games. This paper presents theorems for testing hexadecimal games for periodicity and arithmetic-periodicity. We note however, that not all hexadecimal games are eventually one or the other. We note that **0.123456789** exhibits a new type of periodicity. Starting with $n = 0$, the first fifteen $\mathcal{G}$-values are 0, 1, 0, 2, 2, 1, 1, 3, 2, 4, 4, 5, 5, 6, 4, and thereafter

$$\mathcal{G}(2m-1) = \mathcal{G}(2m) = m-1, \text{ except } \mathcal{G}(2^k+6) = 2^{k-1}.$$

The $\mathcal{G}$-sequence is essentially arithmetic-periodic but with an infinite number of exceptional values that occur in a geometric fashion. That this is the $\mathcal{G}$-sequence can be found in [10]. Also, **0.2048** has the regularity (called a *ruler* regularity in [8]): if $k > 0$ and $j \in \{1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 18\}$ then $\mathcal{G}(13k + j) = 4k + \mathcal{G}(j)$ unless $j = 2$ and k is of the form $(q+1)2^{m+1} + 2^m + 1$ $(q, m \geqslant 0)$. For $q, m \geqslant 0$, then $\mathcal{G}(13((q+1)2^{m+1} + 2^m + 1) + 2) = 2^{m+3}q + 2^{m+2} + 2$.

The game **0.660060008** now calculated up to 390,000 terms, is showing a tendency to be sapp regular with a period length of 96,640.

That an hexadecimal game *probably* has an increasing $\mathcal{G}$-sequence can be shown heuristically. Let a move allow the removal of i beans and then splitting the heap into three. For a heap of size n, a subset of the options are $\{k, k, n-i-2k \mid k = 1, \ldots, \lfloor (n-i)/2 \rfloor\}$ and the $\mathcal{G}$ values are $\mathcal{G}(k) \oplus \mathcal{G}(k) \oplus \mathcal{G}(n-i-2k) = \mathcal{G}(n-i-2k)$, $k = 1, \ldots, \lfloor (n-i)/2 \rfloor$. That is, every second value in the $\mathcal{G}$-sequence for heaps of size 1 through $n-i-2$ is an excluded value for $\mathcal{G}(n)$. This observation gives a proof of the next result which we leave to the reader.

Lemma 1. *Suppose that $G = \mathbf{0.d}_1\mathbf{d}_2 \ldots \mathbf{d}_t$ is an hexadecimal game and that there exists non-negative integers i, odd, and j, even with $\mathbf{d}_i \geqslant 8$ and $\mathbf{d}_j \geqslant 8$ then*

$$\lim_{n \to \infty} \mathcal{G}(n) = \infty.$$

The tables in [3] show some hexadecimal games with periodic behavior and Kenyon had found that **0.3F** was arithmetic-periodic with period 6 and saltus 3. Austin proved the following theorem:

Theorem 2 (Austin [2]). *Suppose that $G = \mathbf{0.d}_1\mathbf{d}_2 \ldots \mathbf{d}_t$ is an hexadecimal game and that there exists non-negative integers e, $p \geqslant t+2$ with $e \geqslant p$ and $m > 0$ such that*

(1) $\mathcal{G}(i+p) = \mathcal{G}(i) + 2^m$ *for all i, $e < i < e + 7p + t$,*

(2) $\mathcal{G}(i) < 2^m$ *for all $i \leqslant e$,*

(3) $\mathcal{G}(i) < 2^{m+1}$ *for all $i \leqslant e + p$,*

(4) *either there exists* $d_{2v+1}\geqslant 8$ *and* $d_{2w}\geqslant 8$ *and for each* g, $0\leqslant g<2^{m+1}$, *there exists* $i>0$ *such that* $\mathcal{G}(i)=g$, *or there exists* $d_u\geqslant 8$ *and for each* $g>0$, $0\leqslant g<2^{m+1}$, *there exists* $2v+1, 2w\geqslant 0$ *such that* $\mathcal{G}(2v+1)=\mathcal{G}(2w)=g$.

Then for all $i>e$, $\mathcal{G}(i+p)=\mathcal{G}(i)+2^m$.

Austin noted that in order to apply the Theorem one may need to take larger than the minimum values of p and s. For example in the tables in Section 2, **0.BA** had period 1 and saltus 1. However, $t=2$, so that to apply the theorem we need to take $p=3$ and $s=3$.

One of the main approaches taken in analyzing Taking-and-Breaking games is to find theorems that say for some function f

if $\mathcal{G}(n+p)=\mathcal{G}(n)$ (*or* $\mathcal{G}(n+p)=\mathcal{G}(n)+s$) *for all* $e<n<f(n)$ *then* $\mathcal{G}(n+p)=\mathcal{G}(n)$ (*or* $\mathcal{G}(n+p)=\mathcal{G}(n)+s$) *for all* $n>e$.

Such results allow a computer program to generate values and check to see if a purported period has repeated enough times without having to call the operator. In this paper we give two such theorems: Theorem 3 for hexadecimal games which are periodic and Theorem 4 which generalizes Austin's theorem to cover all saltuses but the bounds are greater than those in Austin's Theorem. In Section 1 we give a specialized version of Theorem 4 for games where $\mathcal{G}(Ap+Bq+r)=Bp+r$ for some constants A, B and C. The bounds are considerably reduced. These allow us to prove the arithmetic-periodicity of **0.3F3**; **0.209**; **0.338**; **0.2092**; **0.228**; **0.2282**; **0.608** and **0.6082**. We also extend the tables given in [3]. The web page [11] contains the latest information.

1.1. Games with saltus 0

Games with saltus 0 are periodic games. The next theorem is used as a simple test for periodicity.

Theorem 3. *Let G be an hexadecimal game with* $\mathbf{d}_t$ *a non-zero digit with the largest index t. If* $\mathcal{G}(n+p)=\mathcal{G}(n)$ *for* $e<n\leqslant 3(e+p)+t$ *then the* $\mathcal{G}$*-sequence is periodic with period length p and pre-period length e.*

This is a straightforward generalization of the periodicity result for octal games [3, p. 100].

1.2. Arithmetic-periodic games

Here, we generalize Theorem 2 and derive sufficient conditions to test a $\mathcal{G}$-sequence for arithmetic-periodicity for an arbitrary saltus. The bounds on the number of repetitions is much larger than 7 and depends on a couple of new factors.

We need the following definition for the theorem. For any s, $s=2^k r$, where r is an odd number. Then by the Euler–Fermat Theorem, since $\gcd(r,2)=1$, there are positive integers w and j such that $rw=2^j-1$, where we may assume that j is the least possible.

Theorem 4. *Suppose that* $\mathbf{T}=\mathbf{0}.\mathbf{d}_1\mathbf{d}_2\ldots\mathbf{d}_u$ *is an hexadecimal game and that there exist integers* e, $3p \geqslant u+2$ *and* $s=2^{\gamma-1}+j$ *where* $j<2^{\gamma-1}$ *such that*
1. $\mathscr{G}(i+p)=\mathscr{G}(i)+s$ *for all* i *in the range*

$$e < i < t+\max\{3e+p(20+2^{\gamma+1}+2^{2\gamma+1-k}+2^{3\gamma+3-2k}),$$
$$e+p(1+2^{j+k+3-\gamma})\},$$

2. $\mathscr{G}(i)<s$ *for all* $i \leqslant e$,
3. $\mathscr{G}(i)<2s$ *for all* $i \leqslant e+p$,
4. *either there exist* $\mathbf{d}_{2v+1}, \mathbf{d}_{2v}$ *both of which contain* **8**, *and for each* g, $0 \leqslant g < 2s$, *there exists* $i>0$, *such that* $\mathscr{G}(i)=g$, *or there exists* $\mathbf{d}_t$ *which contains* **8**, *and for each* g, $0 \leqslant g < 2s$, *there exist* $2v+1$, $2w \geqslant 0$ *such that* $\mathscr{G}(2v+1)=\mathscr{G}(2w)=g$.

Then for all $i>e$, $\mathscr{G}(i+p)=\mathscr{G}(i)+s$.

The proof can be found in [10].

As examples, we specifically evaluate the bounds on the coefficient of p given in Theorem 4 for the first few values of s which are not powers of 2.

s	γ	j	k	$20+2^{\gamma+1}+2^{2\gamma+1-k}+2^{3\gamma+3-2k}$	$1+2^{j+k+3-\gamma}$
3	2	2	0	572	9
5	3	4	0	4260	17
6	3	2	1	1124	9
7	3	3	0	4260	9
9	4	6	0	33,332	33
10	4	4	1	8500	17
11	4	10	0	33,332	513
12	4	2	2	2228	9

Fortunately, these bounds can be improved for some games and this is given in the next section.

1.3. Games with $\mathscr{G}(Ax+By+z)=Bx+z$

For some of the games we found with saltus 3 and 5, the first bound given in the previous section can be eliminated. This is because the $\mathscr{G}$-sequence has the form $\mathscr{G}(Ax+By+z)=Bx+z$. For example, **0.3F** has the $\mathscr{G}$-sequence

0, 1, 2, 0, 1, 2, 3, 4, 5, 3, 4, 5, 6, 7, 8, 6, 7, 8...

which can be written as $\mathscr{G}(6x+3y+z)=3x+z$, $x=0,1,2,\ldots$, $y=0,1,2$, $z=0,1,2$.

Now, it must be shown that $Bx+z$ is not an excluded value for $\mathscr{G}(Ax+By+z)$. For some x, y and z suppose that $Bx+z$ were an excluded value for $\mathscr{G}(Ax+By+z)$. We may now assume that x, y and z are such that this is the first time this occurs. There are values α_x, α_y, α_z, β_x, β_y, β_z and γ_x, γ_y, γ_z, such that when we take away t and split the heap into three. From splitting into three heaps we get the following

Table 1
Standard forms for games with solutions in the next tables

Game	Standard form
0.08	**0.111333777F**
0.0A	**0.13137F**
0.0B	**0.130F**
0.0C	**0.1133737F**
0.0E	**0.13377F**
0.0F	**0.137F**
0.29	**0.300F**
0.209	**0.3100F**
0.2092	**0.3103F**
0.228	**0.3300F**
0.2282	**0.3303F**
0.608	**0.3700F**
0.6082	**0.3703F**
0.48, 0.4A	**0.13737F**
0.4C, 0.4E	**0.13777F**
0.8, 0.84	**0.113377F**
0.81, 0.85	**0.10F**
0.82	**0.1337F**
0.88, 0.8C	**0.113377FF**
0.89, 0.8D	**0.10FF**
0.8A, 0.8E	**0.1337FF**
0.8B, 0.8F	**0.13FF**
0.A, 0.A1, 0.A4, 0.A5	**0.30F**
0.A2, 0.A3, 0.A6, 0.A7	**0.33F**
0.A8 ,0.A9, 0.AC, 0.AD	**0.30FF**
0.AA, 0.AB, 0.AE, 0.AF	**0.33FF**
0.C, 0.C2, 0.C4, 0.C6	**0.1377F**
0.C8, 0.CA, 0.CC, 0.CE	**0.1377FF**
0.C9, 0.CB, 0.CD, 0.CF	**0.17FF**
0.E8, 0.E9, 0.EA, 0.EB, O.EC, 0.ED, 0.EF	**0.37FF**

equation:

$$A\alpha_x + B\alpha_y + \alpha_z + A\beta_x + B\beta_y + \beta_z + A\gamma_x + B\gamma_y + \gamma_z + t = Ax + By + z.$$

We will shorten the notation by setting $\Sigma_s = \alpha_s + \beta_s + \gamma_s$ for $s \in \{x, y, z\}$.

This special form allows an improvement over the bounds of the previous theorem.

Theorem 5. *Suppose that* $\mathbf{T} = \mathbf{0.d}_1\mathbf{d}_2\ldots\mathbf{d}_u$ *is an hexadecimal game and that* $3p \geqslant u+2$. *Let* $A = BD$, B *odd*, $0 \leqslant y < D$, $0 \leqslant z < B$.

$1'$. *If* $\mathscr{G}(Ax+By+z) = Bx+z$ *for all* $n = Ax+By+z$, $0 \leqslant n \leqslant B(1+2^{j_B+k_B+3-\gamma})$.

$4'$. *Either there exist* $\mathbf{d}_{2v+1}, \mathbf{d}_{2v}$ *both of which contain* **8**, *and for each* g, $0 \leqslant g < 2s$, *there exists* $i > 0$, *such that* $\mathscr{G}(i) = g$ *or there exists* $\mathbf{d}_t$ *which contains* **8**, *and for each* g, $0 \leqslant g < 2s$, *there exist* $2v+1$, $2w \geqslant 0$ *such that* $\mathscr{G}(2v+1) = \mathscr{G}(2w) = g$.

Table 2
Games with saltus 0

Game	Pre-period length	Period length	Saltus	Nim-values
0.10F	0	2	0	$0\dot{0}\dot{1}$
0.30F	0	2	0	$\dot{0}\dot{1}$
0.33F	0	4	0	$\dot{0}12\dot{3}$
0.B	0	2	0	$\dot{0}\dot{1}$
0.B1	2	2	0	$01\dot{2}\dot{0}$
0.B2	1	4	0	$0\dot{1}02\dot{3}$
0.B3	1	4	0	$0\dot{1}20\dot{3}$
0.B5	2	2	0	$01\dot{2}\dot{0}$
0.B7	1	4	0	$0\dot{1}20\dot{3}$
0.D	0	2	0	$\dot{0}\dot{1}$
0.F	0	2	0	$\dot{0}\dot{1}$
0.F1	3	2	0	$012\dot{1}0\dot{0}$
0.F2	1	4	0	$0\dot{1}02\dot{3}$
0 F3	0	4	0	$\dot{0}12\dot{3}$
0.F5	1	2	0	$0\dot{1}\dot{2}$

$5'$. *The equations*

$$(A-B)\Sigma_x + B\Sigma_y + t + 2d = (A-B)x + By,$$

$$2Dd + t = (D-1)(\Sigma_z - z) + B(y - \Sigma_y)$$

have no solutions; *then* $\mathscr{G}(Ax+By+z) = Bx+z$ *for all* $n = Ax+By+z$.

Again, the proofs can be found at [10].

2. Tables

An hexadecimal game is said to be in *standard form* if $\mathbf{d}_1$ is odd. A game $\mathbf{D} = \mathbf{0}.\mathbf{d}_1\mathbf{d}_2\ldots\mathbf{d}_u$ with $\mathbf{d}_1$ even can be reduced to a game $\mathbf{E} = \mathbf{0}.\mathbf{e}_1\mathbf{e}_2\ldots$ in standard form by the following operations (see [3, p. 100]):

$\mathbf{e}_r$ contains **1** (i.e. is odd) if $\mathbf{d}_{r+1}$ contains **1**,

$\mathbf{e}_r$ contains **3** (i.e. is of the form $4m+3$) if $\mathbf{d}_r$ contains **2**,

$\mathbf{e}_r$ contains **7** (i.e. is of the form $8m+7$) if $\mathbf{d}_{r-1}$ contains **4** and in general $\mathbf{e}_r$ contains $\mathbf{2}^{h+2}-1$ if $\mathbf{d}_{r-h}$ contains $\mathbf{2}^{h+1}$ ($h \geqslant -1$).

If e_1 is not odd then we repeat this procedure as often as is necessary. It is not hard to show that $\mathscr{G}_{\mathbf{E}}(n) = \mathscr{G}_{\mathbf{D}}(n+1)$.

Table 3
Games with saltus 2^i

Game	Pre-period length	Period length	Saltus	Nim-Values
0.10FF	3	4	2	$010\dot{1}22\dot{2}$
0.111333777F	1	4	1	$0\dot{1}11\dot{1}$
0.113377F	1	3	1	$0\dot{1}1\dot{1}$
0.113377FF	0	3	1	$\dot{0}1\dot{1}$
0.13137F	1	2	1	$0\dot{1}\dot{1}$
0.130F	4	2	1	$0110\dot{2}\dot{2}$
0.1133737F	1	3	1	$0\dot{1}1\dot{1}$
0.1337F	1	2	1	$0\dot{1}\dot{1}$
0.13377F	1	2	1	$0\dot{1}\dot{1}$
0.1337FF	0	2	1	$\dot{0}\dot{1}$
0.1373F	1	2	1	$0\dot{1}\dot{1}$
0.1377F	0	2	1	$\dot{0}\dot{1}$
0.1377FF	0	2	1	$\dot{0}\dot{1}$
0.13777F	1	2	1	$0\dot{1}\dot{1}$
0.137F	1	2	1	$0\dot{1}\dot{1}$
0.13FF	7	7	4	$0112233\dot{4}45566\dot{7}$
0.17FF	2	3	2	$01\dot{1}2\dot{2}$
0.1A	5	2	1	$01001\dot{2}\dot{2}$
0.1B	5	2	1	$01100\dot{2}\dot{2}$
0.300F	1	53	16	$0\dot{1}01201012323453434567$ $67897678989ABABCABABC$ $DEFEFDEFE\dot{F}$
0.30FF	3	4	2	$01\dot{1}23\dot{2}$
0.33FF	0	1	1	$\ddot{0}$
0.37FF	0	1	1	$\ddot{0}$
0.9B	8	7	4	$01100223\dot{3}44556\dot{6}$
0.9C	28	36	16	$0100222333666333888777555999$ $\dot{B}BADDDFFEGGGIIIKKK$ $MMCJJJOOONNNLLLPP\dot{H}$
0.9E	4	3	2	$0100\dot{2}2\dot{3}$
0.9F	4	3	2	$0100\dot{2}2\dot{3}$
0.B8	9	7	4	$010102323\dot{4}54567\dot{6}$
0.BA	3	1	1	$010\ddot{2}$
0.BB	4	1	1	$0120\ddot{3}$
0.BC	4	6	4	$0101\dot{2}3245\dot{4}$
0.BE	3	1	1	$010\ddot{2}$
0.BF	4	1	1	$0120\ddot{3}$
0.F8	7	6	4	$0101023\dot{2}3454\dot{5}$
0.FA	3	1	1	$010\dot{2}$
0.FB	5	4	4	$01230\dot{4}$
0.FC	7	5	4	$0101232\dot{4}546\dot{7}$
0.FE	3	1	1	$010\dot{2}$
0.FF	0	1	1	$\dot{0}$

Table 4
Games with saltus other than 0 or 2^i

Game	Pre-period length	Period length	Saltus	Nim-Values
0.3100F	0	9	3	$\dot{0}1201201\dot{2}$
0.338	0	9	3	$\dot{0}1201201\dot{2}$
0.3F	0	6	3	$\dot{0}1201\dot{2}$
0.3F3	0	10	5	$\dot{0}12340123\dot{4}$
0.3103F	0	9	3	$\dot{0}1201201\dot{2}$
0.3300F	0	9	3	$\dot{0}1201201\dot{2}$
0.3303F	0	9	3	$\dot{0}1201201\dot{2}$
0.3700F	0	6	3	$\dot{0}1201\dot{2}$
0.3703F	0	6	3	$\dot{0}1201\dot{2}$

Although we examined all the 2 digit games we only investigated the 3-digit games which are in standard form. Theorem 5 could not be applied directly to some of the 2-digit games only to the standard form of the game. The standard forms are listed in the tables.

Table 1 gives the conversion of games to standard form, Table 2 lists the games with saltus 0, Table 3 lists those games with saltus equal to a power of 2, and Table 4 lists games with other saltuses.

In these tables, the dots signify the beginning and end of the period. The values in one period are increased by the saltus from the previous period. In Table 2, every saltus is 0 so the given values repeat. For example, **0.B1** has pre-period 2, and the period has length 2 and the total sequence is $01202020202\ldots$, where the 2 and 0 alternate. In Tables 3 and 4, the saltus is non-zero so every repetition is accompanied by an increase in the nim-values. For example, **0.10FF** has pre-period 3 and a period of length 4 with saltus 2. The values are $010, 1222, 3444, 5666, \ldots$, where the commas delineate the periods, each is 2 greater than the previous one.

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